MATH2058: Analysis I: Home Test **26 Nov 2021, 5:00pm**

Important Notice:

♣ The answer paper **must be submitted before 27 Nov 2021 at 5:00pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ The answer paper must include your name and student ID.

Answer ALL Questions

1. **(30 points)**

   (a) Let S be a countably infinite bounded subset of $\mathbb{R}$. If we let D be the set of all limit points of S, show that there is a family of infinite subsets of $\mathbb{N}$ indexed by D, say $\mathcal{F}$, that is $\mathcal{F} := \{N_\alpha : \ N_\alpha \text{ is an infinte subset of } \mathbb{N}, \alpha \in D\}$, such that $N_\alpha \cap N_\beta$ is a finite set whenever $\alpha, \beta \in D$ with $\alpha \neq \beta$.

   (b) Using Part (a), show that there is an uncountable family of infinite subsets of $\mathbb{N}$, say $\{N_i : i \in I\}$ where I is an uncountable index set, such that $N_\alpha \cap N_\beta$ is a finite set whenever $\alpha, \beta \in I$ with $\alpha \neq \beta$.

   (c) Let $\mathcal{U}$ be a non-empty collection of subsets of $\mathbb{N}$ which satisfies the following conditions:
   (i) $A \cap B \neq \emptyset$ and $A \cap B \in \mathcal{U}$ whenever $A, B \in \mathcal{U}$.
   (ii) A or $A^c \in \mathcal{U}$ for all subsets A of $\mathbb{N}$.
   Let (x_n) be a bounded sequence of real numbers. Show that there is a number L such that for all $\varepsilon > 0$, we have $\{n \in \mathbb{N} : |x_n - L| < \varepsilon\} \in \mathcal{U}$.
   Is such L unique?

2. **(20 points)**

(a) Let f be a real valued function defined on $[a, b]$. A discontinuous point $c \in [a, b]$ for f is called a jumping point if $f(c) = \lim_{x \to c+} f(x)$ or $f(c) = \lim_{x \to c-} f(x)$.
Suppose that f is continuous on $[a, b]$ except finitely many jumping points. Show that there is a sequence of continuous functions $f_n : [a, b] \longrightarrow \mathbb{R}$, $n = 1, 2, ...$ such that $f(t) = \lim_n f_n(t)$ for all $t \in [a, b]$.

(b) Let g be a real valued function defined on $[a, b]$. Suppose that there is a sequence of continuous functions $g_n : [a, b] \longrightarrow \mathbb{R}$, $n = 1, 2...$ such that $g(t) = \lim_n g_n(t)$ for all $t \in [a, b]$.
Show that for any $m, M \in \mathbb{R}$ with $m < M$, there is a sequence of compact sets $(K_n)_{n=1}^{\infty}$ such that $\{t \in [a, b] : m < g(t) < M\} = \bigcup_{n=1}^{\infty} K_n$.

$* * *$ **END OF PAPER** $* * *$